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Algorithm for the Numerical Evaluation of the n-Particle Relativistic Phase Space Integral in Invariant Variables

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A previously derived analytical formulation of the n-particle relativistic momentum phase space integral in invariant variables is simplified. The final equations are eminently suited to machine evaluation and have been incorporated into a Monte Carlo computer program (described elsewhere). The main new feature of this work is the derivation of a set of recurrence relations that allows for the rapid evaluation of dependent invariants in terms of independent ones even in the exceptional regions of the physical region.

1. INTRODUCTION

A problem often encountered in theoretical studies of elementary particle interactions is the numerical evaluation of a momentum phase space integral whose integrand contains the absolute square of some transition matrix element. Another problem, closely related to this evaluation, is the determination of differential cross sections with respect to certain observables of interest. The choice of these observables will quite likely differ from study to study.

The actual evaluation of the multiple phase space integral and of differential cross sections of interest can be done conveniently by a Monte Carlo method in which values of the integration variables (independent variables) are generated randomly according to some specified distribution. Additional variables of posible physical interest (dependent variables) are then computed from the randomly generated independent ones. A useful phase space integration program is then one that satisfies two obvious requirements: (i) event generation should be efficient and (ii) evaluation of any dependent variable should be rapid and accurate.

Over the years much attention [1] has been given to requirement (i). Various algorithms and programs have been developed to increase the efficiency of event generation. Often the programs are tailored to a specific type of physical process, e.g., multiperipheral reactions [2]. Less attention has been given to requirement (ii). The emphasis in the present work is on this second requirement. In fact the method presented here for the generation of *events*, i.e., the random selection of values of the independent variables, is not new; it is identical to that used by Byckling and Kajantie [2]. Nevertheless it is described below because it has a bearing on the approach used to meet requirement (ii).

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Basically in this paper a systematic procedure is presented for the numerical evaluation of the n-particle relativistic momentum phase space integral and the evaluation of numerous dependent quantities using, insofar as possible, invariant Mandelstam variables (taken to be squares of sums of *adjacent* 4-momenta as in Fig. 1). A Monte Carlo computer program based on this algorithm has been written. It is described in detail elsewhere [3].

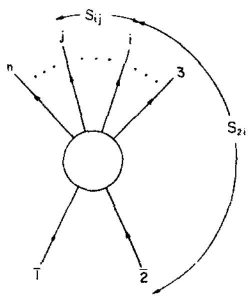

FIG. 1. The n-particle process $\bar{1} + \bar{2} \to 3 + 4 + \cdots + n$ illustrating the ordering of the particles and identifying the invariants.

The groundwork for the present development was laid several years ago [4]. In that paper (hereafter referred to as M) it was shown not only how the phase space integral could be expressed in terms of an independent subset of $3n - 10$ Mandelstam invariants but also how the remaining $(n - 4)(n - 5)/2$ dependent Mandelstam invariants could be expressed in terms of the independent ones. This latter demonstration was nontrivial since the dependent invariants depend on the independent ones in a nonlinear fashion [5].

Similar analytical treatments have been given by Poon [6] and, insofar as the independent invariants are concerned, by Byckling and Kajantie [2]. In the latter case the laboratory 3-momenta of all particles are also calculated, in terms of the independent variables. Presumably any desired dependent quantity is then evaluated from its definition in terms of momentum components in some Lorentz frame. Byckling *et al.* [7] have further developed their approach into a computer subprogram meant to be incorporated into the CERN program FOWL [8]. The distinction between this approach and the present one is that laboratory 3-momenta are used in the former while dependent Mandelstam invariants are used in the latter as intermediary quantities between the generation of independent variables and the evaluation of variables of physical interest. (Of course, the 3-momenta and the independent Mandelstam invariants may themselves be of physical interst.)

In attempting to develop a computer program based on the earlier phase space work [4] it was found that the nonlinear expressions relating dependent to independent invariants led to large numerical inaccuracies for certain values of the independent invariants. These values turn out to lie close to the exceptional regions [9] of the physical region. Such regions are characterized by 4-momenta, which are generally linearly independent, becoming linearly dependent, e.g., the forward direction in two-particle to two-particle scattering. The main new contribution of the present paper is to describe a compact iterative procedure for evaluating the dependent invariants which is useful *even* in the exceptional regions.

As should be clear from the foregoing the variables chosen here to characterize a multiparticle event are invariants. Reasons for choosing these, at least as far as event generation is concerned, have been given by Byckling and Kajantie [2]. Essentially these variables, and most particularly the invariant 4-momenta transfers, are useful in high-energy collisions because they allow for convenient generation of events in those regions of phase space of physical importance—where transverse momenta are small. Thus the probram to be described is directed mainly at these processes. Furthermore it is in precisely these processes that physical events predominate close to the exceptional regions. Even within this context additional advantages accrue from the use of invariant variables, stemming basically from the versatility inherent in *invariant* variables, that is, variables that do not refer to any specific frame. For once a randomly generated event has been labeled by values of all $(3n - 10) + (n - 4)(n - 5)/2 = n(n - 3)/2$ Mandelstam variables these variables can be used to calculate values of physical observables. For example, the momenta components of any or all of the participating particles can be determined in *any* Lorentz frame directly from the invariants. Also, Toller angles which are involved in multi-Regge amplitudes [10] can be found easily thus allowing differential cross sections with respect to these angles to be determined. All of these quantities depend on the dependent as well as independent Mandelstam variables. Another use of the dependent Mandelstam variables arises when multiparticle amplitudes are symmetrized with respect to identical outgoing particles. The symmetrized amplitude will depend on dependent invariants even if the initially unsymmetrized amplitude involves only independent ones.

It is apparent then that knowledge of the values of all Mandelstam variables allows great flexibility in the calculation of physical observables. It is of prime importance that the computation of these variables be fast and accurate. The description of a method for doing this is the main aim of the present work. Program run characteristics are given in [3].

In Sections 2 and 3 the analytical work of M is summarized in a new notation; Section 4 and Appendices A–D contain the new material. Readers interested only in the final phase space equations are directed to Section 5, where a guide to the text is presented. The final Appendix E contains examples of quantities of physical interest expressed in terms of invariant variables.

2. Phase Space Integral

In scattering and production reactions of relativistic elementary particles both dynamic and kinematic effects enter to determine the momenta distribution of the outgoing particles. The dynamic effects are described theoretically by a transition matrix element while the kinematic effects are governed by the available volume in momentum space (phase space).

Consider a process in which two particles of 4-momenta $\bar{p}_1$ and $\bar{p}_2$ collide to produce $n - 2$ outgoing particles of 4-momenta p_i ($i = 3, 4,..., n$) each with mass m_i such that $p_i^2 = m_i^2$, and with the particles labeled cyclically as in Fig. 1 (for later convenience when referring to the multiperipheral model). Then, following Bjorken and Drell [11], the invariant causal amplitude $M_{\beta\alpha}$ is given in terms of the S-matrix by

$$S_{\beta\alpha} = \delta_{\beta\alpha} - i(2\pi)^4\, \delta\left(\sum_{j=3}^{n} p_j - \bar{p}_1 - \bar{p}_2\right)\left\{\prod_{j=1}^{n} [(2\pi)^3\, 2E_j]^{-1/2}\right\} M_{\beta\alpha} \tag{1}$$

with $E_j \equiv (\mathbf{p}_j^2 + m_j^2)^{1/2}$ and where spinless particles are assumed for convenience.

The total cross section for the process is given by [11, 12]

$$\sigma = [2(2\pi)^{3n-10}\, \lambda(s, m_1^2, m_2^2)^{1/2}]^{-1} \left[\prod_{i=3}^{n} \int d^4p_i\, \delta(p_i^2 - m_i^2)\, \theta(p_i^o)\right]$$
$$\times\, \delta\left(\sum_{j=3}^{n} p_j - \bar{p}_1 - \bar{p}_2\right) |\, M_{\beta\alpha}\,|^2 \tag{2}$$

with $s \equiv (\bar{p}_1 + \bar{p}_2)^2$, $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$, and off-shell 4-momenta defined by $p_j = (p_j^o, \mathbf{p}_j)$.

Preparatory to putting (2) entirely in terms of invariant variables define $p_1 \equiv -\bar{p}_1$ and $p_2 \equiv -\bar{p}_2$ so that all momenta are outgoing for convenience and conservation of 4-momentum reads $\sum_{j=1}^{n} p_j = 0$. Then define the invariants (see Fig. 1)

$$s_{ij} \equiv p_{ij}^2 \equiv (p_i + p_{i+1} + \cdots + p_j)^2 \qquad \text{for} \quad 1 \leqslant i \leqslant j \leqslant n. \tag{3}$$

Because of mass-shell and conservation of momentum constraints $s_{1i} = s_{i+1n}$ ($i = 2, 3,..., n - 2$), $s_{ii} = m_i^2$ ($i = 1, 2,..., n$), $s_{1n} = 0$, $s_{1n-1} = m_n^2$, $s_{2n} = m_1^2$, and so there are only $n(n - 3)/2$ invariant variables in the set (3). Of course because of the fact that momenta are four-dimensional $(n - 4)(n - 5)/2$ of these are dependent on the remaining $3n - 10$ independent variables [5], exactly which ones are independent is a matter of choice. These s_{ij} are the invariants dealt with in the present work. All other invariants formed from squares of sums of 4-momenta, such as $(p_1 + p_3 + p_7)^2$, can be expressed in terms of the s_{ij} in a linear fashion [5].

In M it was shown both how the Byers and Yang [13] reformulation of (2) and how the conditions delimiting the physical region could be put entirely in terms of the s_{ij} invariants. In that paper Cayley determinants were used and the reaction studied was $\bar{n} + \bar{1} \rightarrow 2 + 3 + \cdots + (n - 1)$. These results will be transcribed to the present reaction using a slightly different notation. First define

$$L \equiv 2^{n-1}\Delta(1, 2,..., n - 1) \equiv 2^{n-1}\Delta(p_{11}, p_{12},..., p_{1n-1})$$

where the RHS, the Gram determinant of the 4-vectors $p_{11}, p_{12},..., p_{1n-1}$, has $p_{1i} \cdot p_{1j}$ as its i, jth element. Consequently L can be viewed as the determinant of a symmetric matrix whose i, jth element is

$$\begin{aligned} y_{ij} \equiv 2p_{1i} \cdot p_{1j} &= s_{1i} + s_{1j} - s_{i+1j} \qquad \text{for } i < j \\ &= 2s_{1i} \qquad \text{for } i = j. \end{aligned} \tag{4}$$

By analogy to M let $L(i, j,..., k)$ denote the principal minor of L with all rows and columns except $i, j,..., k$ deleted and let $V(i, j,..., k)_{lm}$ denote the cofactor (signed minor) of the element y_{lm} of $L(i, j,..., k)$. The relation of these determinants to the Cayley determinants used in M is simple:

$$G(0, i, j,..., k, n) = (-1)^{\epsilon+1} L(i, j,..., k),$$
$$F(0, i, j,..., k, n)_{lm} = (-1)^{\epsilon} V(i, j,..., k)_{lm},$$

where ϵ is the number of rows (or columns) in $L(i, j,..., k)$.

Once these transcriptions have been carried out on the relevant equations in M a cyclic permutation $1 \to 2$, $2 \to 3$,..., $n-1 \to 1$ must be made on all indices in all resulting determinants to adapt these determinants to the reaction of present interest: $\bar{1} + \bar{2} \to 3 + 4 + \cdots + n$. The reformulation of (2), as given by Eq. (22) of M for $n \geqslant 5$, is then [14]

$$\sigma = [2^{2n-6}(2\pi)^{3n-11}\,\lambda(s_{12}, m_1^2, m_2^2)]^{-1} \left\{\left[\prod_{i=2}^{n-3} \int ds_{1i+1}\right]\left[\prod_{i=2}^{n-2} \int ds_{2i+1}\right] \times \left[\prod_{i=2}^{n-3} \int ds_{i+1i+2}[-L(1, i, i+1, i+2)]^{-1/2}\right]\right\} \sum |M_{\beta\alpha}|^2 \qquad (5)$$

where $\sum |M_{\beta\alpha}|^2 \equiv$ sum of $|M_{\beta\alpha}|^2$ evaluated at all allowed values of the dependent invariants for fixed values of the independent ones. There are 2^{n-5} terms in this sum as will be shown below.

Furthermore the physical region conditions, which determine the ranges of the integration variables in Eq. (5), are obtained from Eqs. (12) and (17) of M [15]:

$L(i, i+1) \leqslant 0$ involving s_{1i} and s_{1i+1} for $i = 1, 2,..., n-2$, (6a)

$L(1, i, i+1) \geqslant 0$ limiting s_{2i+1} for $i = 2, 3,..., n-2$, (6b)

$L(1, i, i+1, i+2) \leqslant 0$ limiting s_{i+1i+2} for $i = 2, 3,..., n-3$, (6c)

$L(1, i, i+1, i+2, i+3) = 0$ determining s_{i+1i+3} for $i = 2, 3,..., n-4$, (6d)

$V(1, i, i+1, i+2, i+3, j+4)_{ij+4} = 0$ determining s_{i+1j+4}
for $j = 2, 3,..., n-5$;
$i = 2, 3,..., j$ (6e)

3. Physical Region

For ease of comprehension the determinantal conditions (6) are displayed as a pyramid structure in Fig. 2. The lower three rows of the pyramid involve determinants that are functions of the independent invariants only. Beginning with the fourth row of the pyramid and moving up more and more dependent invariants are involved.

Any determinant in this pyramid is itself at the apex of a subpyramid as illustrated

$$V(1,2,3,4,5,n-1)_{2n-1} \qquad = 0$$

$$V(1,2,3,4,5,n-2)_{2n-2}\ V(1,3,4,5,6,n-1)_{3n-1} \qquad = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$V(1,2,3,4,5,7)_{27} \cdots\cdots V(1,n-6,n-5,n-4,n-3,n-1)_{n-6n-1} \qquad = 0$$

$$V(1,2,3,4,5,6)_{26}\ V(1,3,4,5,6,7)_{37} \cdots V(1,n-5,n-4,n-3,n-2,n-1)_{n-5n-1} \qquad = 0$$

$$L(1,2,3,4,5)\ L(1,3,4,5,6)\ L(1,4,5,6,7) \cdots\cdots L(1,n-4,n-3,n-2,n-1) \qquad = 0$$

$$L(1,2,3,4)\ L(1,3,4,5)\ L(1,4,5,6) \cdots\cdots L(1,n-3,n-2,n-1) \qquad \leq 0$$

$$L(1,2,3)\ L(1,3,4)\ L(1,4,5) \cdots\cdots L(1,n-2,n-1) \qquad \geq 0$$

$$L(1,2)\ L(2,3)\ L(3,4) \cdots\cdots L(n-2,n-1) \qquad \leq 0$$

FIG. 2. A pyramid structure attempting to show the interconnection of the physical region conditions given by Eqs. (6) in the text.

for $V(1, i, i+1, i+2, i+3, j+4)_{ij+4}$ in Fig. 3. Furthermore such a determinant depends on *all* invariants that occur in the other determinants lying in its subpyramid, plus one new invariant; in fact, these are the only invariants it depends on.

It is therefore possible to impose conditions (6) in a systematic order such that in progressing from one determinant to the next only one new invariant is involved. A possible order (refer to Fig. 2) is to proceed from left to right across rows 1 (the bottom row), 2, 3, and 4 in that order. To handle the cofactors begin again at the left in row 5 but after treating one of the cofactors in this row move diagonally upwards to the left as far as possible before proceeding to the next cofactor in row 5. The final sequence thus encounters the cofactors at the extreme right in rows 5, 6,..., and ends at the apex of the pyramid. This order will be followed below in solving (6).

At each step in such a sequence the determinantal conditions (6) must be solved, thereby furnishing the *ranges* of the independent invariants and the *values* of the dependent invariants. The necessary expressions for the solutions of (6) are developed in Appendix A and are summarized in Appendix B. Equations (B3) and (B4) in particular will now be used to find the upper (+) and lower (−) integration limits of the independent invariants (6a, b, c) and the values of the dependent invariants (6d, e).

$$V(1,i,i+1,i+2,i+3,j+4)_{ij+4} \qquad = 0$$

$$V(1,i,i+1,i+2,i+3,j+3)_{ij+3}\ V(1,i+1,i+2,i+3,i+4,j+4)_{i+1j+4} \qquad = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$V(1,i,i+1,i+2,i+3,i+5)_{ii+5} \cdots\cdots V(1,j-1,j,j+1,j+2,j+4)_{j-1,j+4} \qquad = 0$$

$$V(1,i,i+1,i+2,i+3,i+4)_{ii+4}\ V(1,i+1,i+2,i+3,i+4,i+5)_{i+1i+5} \cdots V(1,j,j+1,j+2,j+3,j+4)_{jj+4} \qquad = 0$$

$$L(1,i,i+1,i+2,i+3)L(1,i+1,i+2,i+3,i+4) \cdots\cdots L(1,j+1,j+2,j+3,j+4) \qquad = 0$$

$$L(1,i,i+1,i+2)\ L(1,i+1,i+2,i+3) \cdots\cdots L(1,j+1,j+2,j+3)L(1,j+2,j+3,j+4) \qquad \leq 0$$

$$L(1,i,i+1)L(1,i+1,i+2) \cdots\cdots L(1,j+2,j+3)L(1,j+3,j+4) \qquad \geq 0$$

$$L(1,i)L(1,i+1)\ L(1,i+2,i+3) \cdots\cdots L(j+3,j+4) \qquad \leq 0$$

FIG. 3. A subpyramid structure of Fig. 2. The determinant at the apex depends on all invariants occurring in the remaining determinants, plus one more: s_{i+1j+4}.

Conditions $L(i, i+1) < 0$

Note that there are $n-2$ equations (for $i = 1, 2,..., n-2$) involving only $n-3$ invariants $s_{12}, s_{13},..., s_{1n-2}$. How they limit the ranges of the invariants may be seen by a straightforward analysis. Write

$$L(i, i+1) = -[s_{1i+1} - (s_{1i}^{1/2} + m_{i+1})^2][s_{1i+1} - (s_{1i}^{1/2} - m_{i+1})^2] \leqslant 0.$$

For $i = 1$ only s_{12} is involved and the inequality may be satisfied by $s_{12} \geqslant (m_1 + m_2)^2$, not surprisingly since $s_{12}^{1/2}$ is the total cm energy of incident particles $\bar{1}$ and $\bar{2}$.

For $i = n-2$ the only invariant involved is s_{1n-2} and by taking $s_{1n-2} \geqslant (m_{n-1} + m_n)^2$ the inequality is satisfied.

For $i = 2, 3,..., n-3$ the relevant condition for satisfying the inequality is

$$s_{1i+1}^{1/2} \leqslant s_{1i}^{1/2} - m_{i+1} \tag{7}$$

which gives an upper limit to s_{1i+1} in terms of s_{1i}.

By reordering (7), making the replacement $i \to i+1$, and using the resulting inequality as a recurrence relation a *lower* limit to s_{1i+1} (including the case $i = 1$) may be found:

$$s_{1i+1}^{1/2} \geqslant s_{1i+2}^{1/2} + m_{i+2} \geqslant s_{1i+3}^{1/2} + m_{i+3} + m_{i+2} \geqslant \cdots \geqslant \sum_{j=i+2}^{n} m_j .$$

In summary, then, conditions (6a) yield the limits

$$\left.\begin{aligned} s_{1i+1}(+) &= (s_{1i}^{1/2} - m_{i+1})^2 \\ s_{1i+1}(-) &= \Big(\sum_{j=i+2}^{n} m_j\Big)^2 \end{aligned}\right\} \quad \text{for} \quad i = 2, 3,..., n-3 \tag{8}$$

along with $s_{12} \geqslant \max\,[(m_1 + m_2^2), (\sum_{j=3}^{n} m_j)^2]$.

Conditions $L(1, i, i+1) \geqslant 0$

Each of these conditions (for $i = 2, 3,..., n-2$) can be satisfied by restricting the range of a single invariant, s_{2i+1}. Notice that $L(1, i, i+1)$ is quadratic in y_{1i+1} and so via (4) is also quadratic in s_{2i+1}. Thus use of (B4) gives the limits,

$$\begin{aligned} s_{2i+1}(\pm) &= s_{11} + s_{1i+1} - y_{1i+1}(\mp) \\ &= s_{11} + s_{1i+1} - (V(1, i, i+1)_{1i+1o} \mp [L(1, i)L(i, i+1)]^{1/2})/L(i) \\ &= s_{11} + s_{1i+1} - (y_{1i}y_{ii+1} \mp [L(1, i)L(i, i+1)^{1/2})/y_{ii} . \end{aligned} \tag{9}$$

Conditions $L(1, i, i+1, i+2) \leqslant 0$

These conditions (for $i = 2, 3, \ldots, n-3$) are satisfied in much the same manner as conditions (6b), by restricting the ranges of the s_{i+1i+2}. Indeed, using (B4) the limits are:

$$\begin{aligned} s_{i+1i+2}(\pm) &= s_{1i} + s_{1i+2} - y_{ii+2}(\mp) \\ &= s_{1i} + s_{1i+2} - (V(1, i, i+1, i+2)_{ii+2o} \\ &\quad \pm [L(1, i, i+1) L(1, i+1, i+2)]^{1/2}) / L(1, i+1). \end{aligned} \tag{10}$$

Conditions $L(1, i, i+1, i+2, i+3) = 0$

By again using (B4) two allowed values of each of the dependent invariants s_{i+1i+3} (for $i = 2, 3, \ldots, n-4$) can be found:

$$\begin{aligned} s_{i+1i+3}(\pm) &= s_{1i} + s_{1i+3} - y_{ii+3}(\mp) \\ &= s_{1i} + s_{1i+3} - (V(1, i, i+1, i+2, i+3)_{ii+3o} \\ &\quad \mp [L(1, i, i+1, i+2) L(1, i+1, i+2, i+3)]^{1/2}) / L(1, i+1, i+2). \end{aligned} \tag{11}$$

This dichotomy, corresponding to two distinctly different configurations of momenta [4], is the origin of the 2^{n-5}-fold summation in (5). That is, $\sum | M_{\beta\alpha} |^2 =$ sum of $| M_{\beta\alpha} |^2$ evaluated at all of the values of s_{i+1i+3} given by (11), and the following values of s_{i+1j+4}.

Conditions $V(1, i, i+1, i+2, i+3, j+4)_{ij+4} = 0$

Here (B3) is used to find the allowed values of the dependent invariants s_{i+1j+4} (for $j = 2, 3, \ldots, n-5$; $i = 2, 3, \ldots, j$):

$$\begin{aligned} s_{i+1j+4} &= s_{1i} + s_{1j+4} - y_{ij+4} \\ &= s_{1i} + s_{1j+4} \\ &\quad - V(1, i, i+1, i+2, i+3, j+4)_{ij+4o} / L(1, i+1, i+2, i+3). \end{aligned} \tag{12}$$

The cofactors on the RHS of (11) and (12) may be evaluated in a systematic fashion using (B1). Indeed, note that

$$V(1, i, i+1, i+2, i+3)_{ii+3o} = -\sum_k V(1, i+1, i+2, i+3)_{ki+3}\, y_{ik}$$

$$\text{with} \quad k = 1, i+1, i+2,$$

$$\begin{aligned} V(1, i, i+1, i+2, i+3, j+4)_{ij+4o} &= -\sum_k V(1, i+1, i+2, i+3, j+4)_{kj+4}\, y_{ik} \\ &= \sum_{k,l} V(1, i+1, i+2, i+3)_{kl}\, y_{ik} y_{lj+4} \end{aligned}$$

$$\text{with} \quad k, l = 1, i+1, i+2, i+3.$$

Thus if the basic cofactors (3×3 determinants) $V(1, i+1, i+2, i+3)_{kl}$ with $i = 2, 3, \ldots, n-4$ and $k, l = 1, i+1, i+2, i+3$ are known all the required cofactors of larger dimensions may be generated from them in a sequential manner. Furthermore the basic ones are functions only of the independent invariants and can be evaluated once the latter quantities are known. Unfortunately difficulties stemming from exceptional regions of the physical region preclude using this method for evaluating (11) and (12). Section 4 is devoted to a careful tratment of this problem.

4. Exceptional Regions and the Dependent Invariants

At this stage in the development the cross section is given by (5) together with (8)–(10) which specify the ranges of the independent invariants and with (11)–(12) which specify the dependent invariants in terms of the independent ones. All of this formal work stems from an earlier paper [4] although a different notation was used there. Similar treatments have been given by Poon [6] and, with the exception of Eqs. (11)–(12), by Byckling and Kajantie [2]. No one has as yet directly handled the difficulties caused by the exceptional regions.

With the present choice of independent invariants the exceptional regions of concern are those where some $L(1, i, i+1) \to 0$ which can occur by having s_{2i+1} approach either of its limiting values, $s_{2i+1}(\pm)$ of (9). This results in the nominally independent invariants s_{ii+1} and s_{i+1i+2} becoming dependent invariants for by (10) their ranges shrink to zero and, furthermore, cause $L(1, i-1, i, i+1)$ and $L(1, i, i+1, i+2)$ to vanish as can be seen by using (B5). A more serious difficulty arises when solving $L(1, i-1, i, i+1, i+2) = 0$ for s_{ii+2} because in (11) (with $i \to i-1$) both numerator and denominator vanish in the exceptional region. Similarly, several invariants given by (12) are also defined by expressions that have vanishing numerators and denominators—those that involve $L(1, i-1, i, i+1)$ or $L(1, i, i+1, i+2)$ in the denominator—in the exceptional regions. These 0/0 situations result in large inaccuracies in machine computations and a method for avoiding such numerical disasters is presented below. Such occurrences are quite common when modeling peripheral reactions. For in these reactions the matrix element may typically contain a factor exp $(A\, s_{2i+1})$, with $A > 0$, thus enhancing the importance of generated events for which the value of the 4-momentum transfer s_{2i+1} is near its upper limit.

A successful resolution of these difficulties begins by replacing the independent invariants s_{i+1i+2} by angles ϕ_{i+1} defined by (compare (10)):

$$s_{i+1i+2} = s_{1i} + s_{1i+2} - (V(1, i, i+1, i+2)_{ii+2o} + \cos \phi_{i+1}[L(1, i, i+1)L(1, i+1, i+2)]^{1/2})/L(1, i+1). \quad (13)$$

Then according to (B2) it follows that

$$V(1, i, i+1, i+2)_{ii+2} = -\cos \phi_{i+1}[L(1, i, i+1)L(1, i+1, i+2)]^{1/2} \quad (14)$$

and from (B5) that

$$L(1, i, i+1, i+2) = \sin^2 \phi_{i+1}[L(1, i, i+1)L(1, i+1, i+2)]/L(1, i+1). \quad (15)$$

A direct benefit [2, 4] of this change of variable is that the denominator factor $[-L(1, i, i+1, i+2)]^{1/2}$ in (5), which vanishes at the limits of integration, is removed. For, by use of (13) and (15)

$$\int ds_{i+1\,i+2}[-L(1, i, i+1, i+2)]^{-1/2} = \int d\phi_{i+1}[-L(1, i+1)]^{-1/2} \tag{16}$$

with $\phi_{i+1}(-) = 0$ and $\phi_{i+1}(+) = \pi$.

The angle ϕ_{i+1} can be interpreted geometrically [4] as the azimuthal angle of the vector $\mathbf{p}_{i+3n}$ $(= -\mathbf{p}_{i+2})$ in the Lorentz frame where $\mathbf{p}_{i+2n} = 0$, where $\bar{\mathbf{p}}_1$ $(= -\mathbf{p}_1)$ is along the $+z$ direction, and where $\mathbf{p}_{i+1}$ $(= \mathbf{p}_{i+1n})$ lies in the x–z plane with positive x component. In other words

$$\cos \phi_{i+1} = (\mathbf{p}_1 \times \mathbf{p}_{i+3n}) \cdot (\mathbf{p}_1 \times \mathbf{p}_{i+1}) / \mid \mathbf{p}_1 \times \mathbf{p}_{i+3n} \mid \mid \mathbf{p}_1 \times \mathbf{p}_{i+1} \mid$$

with $\mathbf{p}_{i+2n} = 0$. This is explored more fully in Appendix E.

In a similar fashion dichotomic variables r_{i+1}, which can take as only the values $+1$ and -1, are introduced (compare (11):

$$\begin{aligned} s_{i+1i+3} = s_{1i} + s_{1i+3} - (&V(1, i, i+1, i+2, i+3)_{ii+3o} \\ &+ r_{i+1}[L(1, i, i+1, i+2)L(1, i+1, i+2, i+3)]^{1/2})/L(1, i+1, i+2) \end{aligned} \tag{17}$$

leading to

$$\begin{aligned} &V(1, i, i+1, i+2, i+3)_{ii+3} \\ &\quad = -r_{i+1}[L(1, i, i+1, i+2)L(1, i+1, i+2, i+3)]^{1/2}. \end{aligned} \tag{18}$$

The reason for introducing the quantities ϕ_{i+1} and r_{i+1} will become clearer below. In passing, it may be noticed from (13) that even when $L(1, i, i+1) \to 0$ and the range of s_{i+1i+2} shrinks to zero the angle ϕ_{i+1} remains unrestricted and thus retains its status as an independent variable. The dependent invariants which come from solving (6e) will ultimately involve ϕ_{i+1} and r_{i+1} rather than s_{i+1i+2} and s_{i+1i+3}, respectively. Now, as a preliminary step in handling Eqs. (6e), define a set of modified cofactors:

$$W(1, i, i+1, i+2, j+4)_{ij+4} \equiv \frac{V(1, i, i+1, i+2, j+4)_{ij+4}}{[-L(1, i+1, i+2)\, L(1, i, i+1, i+2)]^{1/2}}, \tag{19a}$$

$$W(1, i+1, i+2, j+4)_{i+1j+4} \equiv \frac{V(1, i+1, i+2, j+4)_{i+1j+4}}{[-L(1, i+2)\, L(1, i+1, i+2)]^{1/2}}, \tag{19b}$$

$$W(1, i+2, j+4)_{i+2j+4} \equiv \frac{V(1, i+2, j+4)_{i+2j+4}}{[-L(1)\, L(1, i+2)]^{1/2}}. \tag{19c}$$

In Appendix C the following recurrence relations are established for the modified cofactors of (19) for $j = 2, 3, \ldots, n-5$ and $i = j, j-1, \ldots, 2$:

$$\begin{aligned} &W(1, i, i+1, i+2, j+4)_{ij+4} \\ &\quad = -r_{i+1}\Big\{-\cos\phi_{i+2} W(1, i+1, i+2, i+3, j+4)_{i+1j+4} \\ &\qquad + \frac{V(1, i+2, i+3)_{i+2i+3}}{[L(1, i+2)\, L(1, i+3)]^{1/2}} \sin\phi_{i+2} W(1, i+2, i+3, j+4)_{i+2j+4} \\ &\qquad + \Big[\frac{L(1)\, L(1, i+2, i+3)}{L(1, i+2)\, L(1, i+3)}\Big]^{1/2} \sin\phi_{i+2} W(1, i+3, j+4)_{i+3j+4}\Big\}; \end{aligned} \tag{20a}$$

$$\begin{aligned} &W(1, i+1, i+2, j+4)_{i+1j+4} \\ &\quad = -\sin\phi_{i+2} W(1, i+1, i+2, i+3, j+4)_{i+1j+4} \\ &\qquad - \frac{V(1, i+2, i+3)_{i+2i+3}}{[L(1, i+2)\, L(1, i+3)]^{1/2}} \cos\phi_{i+2} W(1, i+2, i+3, j+4)_{i+2j+4} \\ &\qquad - \Big[\frac{L(1)\, L(1, i+2, i+3)}{L(1, i+2)\, L(1, i+3)}\Big]^{1/2} \cos\phi_{i+2} W(1, i+3, j+4)_{i+3j+4}; \end{aligned} \tag{20b}$$

$$\begin{aligned} &W(1, i+2, j+4)_{i+2j+4} \\ &\quad = -\Big[\frac{L(1)\, L(1, i+2, i+3)}{L(1, i+2)\, L(1, i+3)}\Big]^{1/2} W(1, i+2, i+3, j+4)_{i+2j+4} \\ &\qquad + \frac{V(1, i+2, i+3)_{i+2i+3}}{[L(1, i+2)\, L(1, i+3)]^{1/2}} W(1, i+3, j+4)_{i+3j+4}. \end{aligned} \tag{20c}$$

Equations (20) can be used as recurrence relations on i to calculate $W(1, i, i+1, i+2, j+4)_{ij+4}$, $W(1, i+1, i+2, j+4)_{i+1j+4}$, and $W(1, i+2, j+4)_{i+2j+4}$ beginning with $i = j$. Initial values for the modified cofactors on the RHS of (20) are therefore needed. They can be obtained by using (14), (15), and (18) in (19):

$$\begin{aligned} &W(1, j+1, j+2, j+3, j+4)_{j+1j+4} \\ &\quad = -\Big[-\frac{L(1, j+4)}{L(1)}\Big]^{1/2} \Big[\frac{L(1)\, L(1, j+3, j+4)}{L(1, j+3)\, L(1, j+4)}\Big]^{1/2} \sin\phi_{j+3} r_{j+2}; \end{aligned} \tag{21a}$$

$$\begin{aligned} &W(1, j+2, j+3, j+4)_{j+2j+4} \\ &\quad = -\Big[-\frac{L(1, j+4)}{L(1)}\Big]^{1/2} \Big[\frac{L(1)\, L(1, j+3, j+4)}{L(1, j+3)\, L(1, j+4)}\Big]^{1/2} \cos\phi_{j+3}; \end{aligned} \tag{21b}$$

$$\begin{aligned} &W(1, j+3, j+4)_{j+3j+4} \\ &\quad = \Big[-\frac{L(1, j+4)}{L(1)}\Big]^{1/2} \frac{V(1, j+3, j+4)_{j+3j+4}}{[L(1, j+3)\, L(1, j+4)]^{1/2}}. \end{aligned} \tag{21c}$$

Finally (12), (13), and (17) may be rewritten in terms of the modified cofactors of (19). These manipulations are carried out in Appendix D with the results.

$$s_{i+1i+2} = s_{1i} + s_{1i+2} - \frac{V(1, i, i+2)_{ii+2o}}{L(1)} + \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \overline{W}(1, i, i+2)_{ii+2}$$

$$+ \cos \phi_{i+1} \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \left[-\frac{L(1, i+2)}{L(1)}\right]^{1/2} \left[\frac{L(1)\, L(1, i, i+1)}{L(1, i)\, L(1, i+1)}\right]^{1/2}$$

$$\times \left[\frac{L(1)\, L(1, i+1, i+2)}{L(1, i+1)\, L(1, i+2)}\right]^{1/2} \quad \text{for} \quad i = 2, 3,\ldots, n-3; \tag{22}$$

$$s_{i+1i+3} = s_{1i} + s_{1i+3} - \frac{V(1, i, i+3)_{ii+3o}}{L(1)} + \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \Big\{\overline{W}(1, i, i+3)_{ii+3}$$

$$- \left[\frac{L(1)\, L(1, i, i+1)}{L(1, i)\, L(1, i+1)}\right]^{1/2} \overline{W}(1, i, i+1, i+3)_{ii+3}\Big\}$$

$$- r_{i+1} \sin \phi_{i+1} \sin \phi_{i+2} \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \left[-\frac{L(1, i+3)}{L(1)}\right]^{1/2}$$

$$\times \left[\frac{L(1)\, L(1, i, i+1)}{L(1, i)\, L(1, i+1)}\right]^{1/2} \left[\frac{L(1)\, L(1, i+2, i+3)}{L(1, i+2)\, L(1, i+3)}\right]^{1/2}$$

$$\text{for} \quad i = 2, 3,\ldots, n-4; \tag{23}$$

$$s_{i+1j+4} = s_{1i} + s_{1j+4} - \frac{V(1, i, j+4)_{ij+4o}}{L(1)} + \left[-\frac{L(1, i)}{L(1)}\right]^{1/2}$$

$$\times \Big\{\overline{W}(1, i, j+4)_{ij+4} - \left[\frac{L(1)\, L(1, i, i+1)}{L(1, i)\, L(1, i+1)}\right]^{1/2} [\overline{W}(1, i, i+1, j+4)_{ij+4}$$

$$- \sin \phi_{i+1} W(1, i, i+1, i+2, j+4)_{ij+4}]\Big\}$$

$$\text{for} \quad j = 2, 3,\ldots, n-5; \quad i = j, j-1,\ldots, 2. \tag{24}$$

The bar over the modified cofactors in these last three equations signifies that the cofactor is evaluated subject to the condition that the next higher order cofactor vanish. Specifically

$$\overline{W}(1, i, k)_{ik} \equiv W(1, i, k)_{ik} \quad \text{subject to } W(1, i, i+1, k)_{ik} = 0,$$

$$\overline{W}(1, i, i+1, k)_{ik} \equiv W(1, i, i+1, k)_{ik} \quad \text{subject to } W(1, i, i+1, i+2, k)_{ik} = 0.$$

The imposition of these conditions is very easy if Eqs. (20), extended to include $i, j = 0, 1$ are used to evaluate the barred modified cofactors: the first term on the RHS of both (20b) and (20c) is set equal to zero. Relations (20) can thereby be used with $j = 0, 1, 2,\ldots, n-5$ and $i = j, j-1,\ldots, 0$ when evaluating (22)–(24).

The scheme as outlined in this Section for calculating s_{i+1i+2}, s_{i+1i+3}, and s_{i+1j+4}, although not too transparent, does resolve the earlier 0/0 numerical difficulties associated with the exceptional regions. For notice that the determinants $L(1, i, i+1)$ whose vanishing caused the trouble in (10), (11), and (12) no longer appear in crucial positions; they were suppressed by use of the modified cofactors (19). The evaluations in (22), (23), and (24) are thus uneventful and are straightforward to implement on a computer.

As a final point notice that several determinantal factors keep recurring in the above equations as coefficients of the modified cofactors. These factors are functions of the invariants that occur in the lower two rows only of the physical region pyramid of Fig. 2. In totality they are:

$$\left[-\frac{L(1,i)}{L(1)}\right]^{1/2} \qquad \text{for} \quad i = 2, 3, \ldots, n-1, \tag{25a}$$

$$\left[\frac{L(1)\, L(1,i,i+1)}{L(1,i)\, L(1,i+)1}\right]^{1/2} \qquad \text{for} \quad i = 2, 3, \ldots, n-2, \tag{25b}$$

$$\frac{V(1,i,i+1)_{ii+1}}{[L(1,i)\, L(1,i+1)]^{1/2}} \qquad \text{for} \quad i = 2, 3, \ldots, n-2. \tag{25c}$$

5. Summary

A guide to the final equations suitable for the numerical evaluation of the n-particle phase space integral and computation of the nonlinear dependent invariants is as follows.

The expression for the total cross section, valid for $n \geqslant 4$, is (from (5)

$$\sigma = [2^{n-1}(2\pi)^{3n-11}\, \lambda(s_{12}, m_1^2, m_2^2)]^{-1} \left\{ \left[\prod_{i=2}^{n-3} \int ds_{1i+1}\right] \left[\prod_{i=2}^{n-2} \int ds_{2i+1}\right] \times \left[\prod_{i=2}^{n-3} \int_0^\pi d\phi_{i+1}[-L(1, i+1)]^{-1}\right] \right\} |\, M_{\beta\alpha} \,|^2_{\text{avg}} \tag{26}$$

where $|\, M_{\beta\alpha}\,|^2_{\text{avg}} = |\, M_{\beta\alpha}\,|^2$ *averaged* over all allowed values of the dependent invariants (if any) for fixed values of the independent ones and where the limits of integration are given by (8) for s_{1i+1} and by (9) for s_{2i+1}.

The evaluation of the independent invariants s_{i+1i+2}, which were replaced by angles ϕ_{i+1}, (for $i = 2, 3, \ldots, n-3$) and the dependent invariants s_{i+1i+3} (for $i = 2, 3, \ldots, n-4$) and s_{i+1j+4} (for $j = 2, 3, \ldots, n-5$ and $i = j, j-1, \ldots, 2$) is more involved. The procedure is to use (20) as recurrence relations on i initialized by (21) to systematically find the modified cofactors W and $\overline{W}$ which in turn are used in (22), (23), and (24) to obtain the remaining nonlinear invariants. Notice that the meaning of $\overline{W}$ is spelled out in the paragraph following (24) and that r_{i+1}, which is involved in the definition of s_{i+1i+3}, takes on only the values $+1$ and -1.

As mentioned in the Introduction these equations have all been successfully incorporated into a Monte Carlo numerical integration program [3].

APPENDIX A

Several determinantal identities and other useful properties of determinants are presented here. Some of these can be found in earlier works [4, 16] but this Appendix contains the essential features of the derivations. The results are summarized in Appendix B. For ease of manipulation the notation used in this Appendix A differs from that used elsewhere in the paper.

Consider a square matrix (x_{ab}) with determinant $X \equiv \det(x_{ab})$. Let $X_{ab\ldots c}$ denote the principal minor of X with rows and columns $a, b, \ldots, c$ *deleted* and let $Y(a, b, \ldots, c)_{fg}$ denote the cofactor (signed minor) of the element x_{fg} of $X_{ab\ldots c}$.

By an expansion due to Cauchy [17]

$$X = -\sum_{a,c\neq b} Y(b)_{ac}\, x_{ab}x_{bc} + X_b x_{bb}\,. \tag{A1}$$

But by an expansion of X in terms of the elements of the bth column

$$X = \sum_{a\neq b} Y_{ab}x_{ab} + X_b x_{bb}\,. \tag{A2a}$$

Equating these two expressions for X yields

$$Y_{ab} = -\sum_{c\neq b} Y(b)_{ac}\, x_{bc} \tag{A3a}$$

$$= -\sum_{c\neq a,b} Y(b)_{ac}\, x_{bc} - X_{ab}x_{ba} \tag{A4a}$$

$$\equiv Y_{abo} - X_{ab}x_{ba} \tag{A5a}$$

where Y_{abo} is Y_{ab} evaluated at $x_{ba} = 0$.

Similarly, by expanding X in terms of the elements of the bth row

$$X = \sum_{c\neq b} Y_{bc}x_{bc} + X_b x_{bb} \tag{A2b}$$

and equating this expansion to that in (A1) yields

$$Y_{bc} = -\sum_{a\neq b} Y(b)_{ac}\, x_{ab} \tag{A3b}$$

$$= -\sum_{a\neq b,c} Y(b)_{ac}\, x_{ab} - X_{bc}x_{cb} \tag{A4b}$$

$$= Y_{bco} - X_{bc}x_{cb}\,. \tag{A5b}$$

Of course, for a symmetric matrix (x_{ab}) there is no distinction between the sets of equations (A2a)–(A5a) and (A2b)–(A5b).

Several very important identities to be displayed shortly are based on the Jacobi ratio theorem [18]. When applied to the matrix (x_{ab}) the theorem states that

$$[\mathrm{adj}(x_{ab})]^{(k)} = X^{k-1} \cdot \mathrm{adj}^{(k)}(x_{ab})$$

where:

(i) $\mathrm{adj}(x_{ab}) \equiv (Y_{ba})$ is the (adjugate) matrix obtained by replacing the elements of (x_{ab}) by their cofactors Y_{ab} and transposing the result.

(ii) $[\mathrm{adj}(x_{ab})]^{(k)}$ is the kth compound [19] of $\mathrm{adj}(x_{ab})$; that is, the matrix whose elements are minors of $\mathrm{adj}(x_{ab})$ of order k.

(iii) $\mathrm{adj}^{(k)}(x_{ab})$ is the kth adjugate compound [19] of (x_{ab}); that is, the transpose of the matrix whose elements are the cofactors in (x_{ab}) of minors of order k.

The desired results follow for the choice $k = 2$. In this case the theorem reads

$$\begin{vmatrix} \mathrm{adj}(x_{ij})_{ac} & \mathrm{adj}(x_{ij})_{ad} \\ \mathrm{adj}(x_{ij})_{bc} & \mathrm{adj}(x_{ij})_{bd} \end{vmatrix} = X \cdot \text{cofactor of} \begin{vmatrix} x_{ca} & x_{cb} \\ x_{da} & x_{db} \end{vmatrix} \text{ in } (x_{ij})$$

or

$$\begin{vmatrix} Y_{ca} & Y_{da} \\ Y_{cb} & Y_{db} \end{vmatrix} = X \cdot Y(Y_{ca})_{db}$$

or

$$Y_{ca} Y_{db} - Y_{da} Y_{cb} = X \cdot Y(Y_{ca})_{db}$$

where $Y(Y_{ca})_{db}$ is the cofactor of the element x_{db} in the cofactor Y_{ca}.

Particularly useful identities result from the choices:

$$\text{(a)} \quad c = a,\ d = b\colon\ X_a X_b - Y_{ba} Y_{ab} = X \cdot X_{ab}\,, \tag{A6}$$

$$\text{(b)} \quad c = a\colon\ X_a Y_{db} - Y_{da} Y_{ab} = X \cdot Y(a)_{db}\,. \tag{A7}$$

As an example of the use of (A6) with a *symmetric* matrix (x_{ab}) substitution of (A5) into (A6) yields an expression for X which is quadratic in $x_{ab} = x_{ba}$:

$$X = -X_{ab} x_{ab}^2 + 2 Y_{abo} x_{ab} + (X_a X_b - Y_{abo}^2)/X_{ab}$$

which may be written as

$$X = -X_{ab}[x_{ab} - x_{ab}(+)][x_{ab} - x_{ab}(-)] \tag{A8}$$

where

$$x_{ab}(\pm) \equiv Y_{abo}/X_{ab} \pm (X_a X_b)^{1/2}/|\, X_{ab} \,|$$

are the solutions of $X = 0$.

Two more identities which will be used extensively in Appendix C follow from (A3)–(A5), (A7). In deriving these it should be noted that the properties derived above for the determinant X can equally well be applied to any principal minor.

Then

$$Y(b)_{ac}\,Y_{cb} = -\sum_{d\neq b} Y(b)_{ac}\,Y(b)_{cd}\,x_{bd} \qquad \text{by (A3a)}$$

$$= -\sum_{d\neq b} [X_{bc}Y(b)_{ad} - X_b Y(bc)_{ad}]\,x_{bd} \qquad \text{by (A7)}$$

$$= X_{bc}Y_{ab} - X_b Y(c)_{ab} \qquad \text{by (A3a)(A9)}$$

and

$$Y(a)_{bc}\,Y(b)_{ca} = \left[\sum_{d\neq a,b} Y(ab)_{dc}\,x_{db}\right]\left[\sum_{e\neq a,b} Y(ab)_{ce}\,x_{ae}\right] \qquad \text{by (A3a, b)}$$

$$= \sum_{d,e\neq a,b} Y(ab)_{dc}\,Y(ab)_{ce}\,x_{db}x_{ae}$$

$$= \sum_{d,e\neq a,b} [X_{abc}Y(ab)_{de} - X_{ab}Y(abc)_{de}]\,x_{db}x_{ae} \qquad \text{by (A7)}$$

$$= \sum_{d\neq a,b} [-X_{abc}Y(b)_{da} + X_{ab}Y(bc)_{da}]\,x_{db} \qquad \text{by (A3a)}$$

$$= X_{abc}[Y_{ba} + X_{ab}x_{ab}] - X_{ab}[Y(c)_{ba} + X_{abc}x_{ab}] \qquad \text{by (A4b)}$$

$$= X_{abc}Y_{ba} - X_{ab}Y(c)_{ba}\,. \qquad \text{(A10)}$$

Appendix B

The important results of Appendix A are summarized below in the notation of the text: L denotes the determinant of a symmetric matrix with elements y_{ab} , $L(a, b,..., c)$ denotes the principal minor of L with rows and columns $a, b,..., c$ *retained*, and $V(a, b,..., c)_{fg}$ denotes the cofactor of the element y_{fg} of $L(a, b,..., c)$. To reduce the number of indices exhibited α will be an inclusive label denoting retained but undisplayed rows and columns of L and its minors. For example $L(a, b,..., c)$ might be written as $L(\alpha)$, $L(\alpha, a, b)$, $L(\alpha, c)$, etc., depending on which indices need not be displayed.

The results are

(a) (from A3)

$$V(\alpha, a, b)_{ab} = -\sum_{c=\alpha,a} V(\alpha, a)_{ac}\,y_{bc}\,, \qquad \text{(B1)}$$

(b) (from A5)

$$V(\alpha, a, b)_{ab} = V(\alpha, a, b)_{abo} - L(\alpha)\,y_{ab}\,, \qquad \text{(B2)}$$

where $V(\alpha, a, b)_{abo} \equiv V(\alpha, a, b)_{ab}$ evaluated at $y_{ab} = 0$.

Thus $V(\alpha, a, b)_{ab} = 0$ may be solved for

$$y_{ab} = V(\alpha, a, b)_{abo}/L(\alpha). \qquad \text{(B3)}$$

(c) (From A8) $L(\alpha, a, b) = 0$ is quadratic in y_{ab} and has the solutions

$$y_{ab}(\pm) \equiv \frac{V(\alpha, a, b)_{abo}}{L(\alpha)} \pm \frac{[L(\alpha, a)\, L(\alpha, b)]^{1/2}}{|\, L(\alpha)|}. \qquad \text{(B4)}$$

Furthermore,

$$L(\alpha, a, b) = -L(\alpha)[y_{ab} - y_{ab}(+)][y_{ab} - y_{ab}(-)]. \qquad \text{(B5)}$$

Finally there are four important identities:

(d) (from A6)

$$V(\alpha, a, b)_{ab}^2 = L(\alpha, a)\, L(\alpha, b) - L(\alpha)\, L(\alpha, a, b); \qquad \text{(B6)}$$

(e) (from A7)

$$V(\alpha, a, b, c)_{ac}\, V(\alpha, a, b, c)_{cb} = L(\alpha, a, b) V(\alpha, a, b, c)_{ab} - L(\alpha, a, b, c) V(\alpha, a, b)_{ab}\,; \qquad \text{(B7)}$$

(f) (from A9)

$$V(\alpha, a, c)_{ac}\, V(\alpha, a, b, c)_{cb} = L(\alpha, a) V(\alpha, a, b, c)_{ab} - L(\alpha, a, c) V(\alpha, a, b)_{ab} \qquad \text{(B8)}$$

(g) (from A10)

$$V(\alpha, b, c)_{bc}\, V(\alpha, a, c)_{ca} = L(\alpha) V(\alpha, a, b, c)_{ab} - L(\alpha, c) V(\alpha, a, b)_{ab}\,. \qquad \text{(B9)}$$

APPENDIX C

The purpose of this Appendix is to derive the recurrence relations of (20); that is, formulas that will allow the iterative evaluation of the cofactors $V(1, i, i+1, i+2, j+4)_{ij+4}$, $V(1, i+1, i+2, j+4)_{i+1j+4}$, and $V(1, i+2, j+4)_{i+2j+4}$. Such expressions will be found by making extensive use of (B8) and (B9).

$V(1, i, i+1, i+2, j+4)_{ij+4}$

By (B9) with $\alpha = 1, i+1, i+2$; $a = i, b = j+4, c = i+3$ and using the physical requirement (6e) that

$$V(1, i, i+1, i+2, i+3, j+4)_{ij+4} = 0 \qquad \text{(C1)}$$

it follows that

$$V(1, i, i+1, i+2, j+4)_{ij+4} = -\frac{V(1, i, i+1, i+2, i+3)_{ii+3}\, V(1, i+1, i+2, i+3, j+4)_{i+3j+4}}{L(1, i+1, i+2, i+3)}. \qquad \text{(C2)}$$

At this stage it need only be *imagined* that (C1) has been solved (for s_{i+1j+4}). Recurrence relations (20), based on (C2) and several following equations, will then yield the actual value of the cofactor on the LHS of (C2). Subsequently, in (24) and Appendix D, this cofactor will be used to finally solve (C1) for s_{i+1j+4}.

By (B8) with $\alpha = 1, i+2$; $a = i+3$, $b = j+4$, and $c = i+1$ it follows that

$$V(1, i+1, i+2, i+3, j+4)_{i+3j+4} = \frac{V(1, i+1, i+2, i+3)_{i+1i+3}\, V(1, i+1, i+2, i+3, j+4)_{i+1j+4}}{L(1, i+2, i+3)} + \frac{L(1, i+1, i+2, i+3)\, V(1, i+2, i+3, j+4)_{i+3j+4}}{L(1, i+2, i+3)}. \tag{C3}$$

By (B8) with $\alpha = 1$; $a = i+3$, $b = j+4$, and $c = i+2$ it follows that

$$V(1, i+2, i+3, j+4)_{i+3j+4} = \frac{V(1, i+2, i+3)_{i+2i+3}\, V(1, i+2, i+3, j+4)_{i+2j+4}}{L(1, i+3)} + \frac{L(1, i+2, i+3)\, V(1, i+3, j+4)_{i+3j+4}}{L(1, i+3)}. \tag{C4}$$

Finally (C4) may be substituted into (C3) and the latter may then be substituted into (C2) to give

$$V(1, i, i+1, i+2, j+4)_{ij+4} = -\frac{\begin{pmatrix} V(1, i+1, i+2, i+3)_{i+1i+3}\, V(1, i, i+1, i+2, i+3)_{ii+3} \\ \times V(1, i+1, i+2, i+3 j, +4)_{i+1j+4} \end{pmatrix}}{L(1, i+2, i+3)\, L(1, i+1, i+2, i+3)} - \frac{\begin{pmatrix} V(1, i+2, i+3)_{i+2i+3}\, V(1, i, i+1, i+2, i+3)_{ii+3} \\ \times V(1, i+2, i+3, j+4)_{i+2j+4} \end{pmatrix}}{L(1, i+3)\, L(1, i+2, i+3)} - \frac{V(1, i, i+1, i+2, i+3)_{ii+3}\, V(1, i+3, j+4)_{i+3j+4}}{L(1, i+3)}. \tag{C5}$$

$V(1, i+1, i+2, j+4)_{i+1j+4}$

By (B9) with $\alpha = 1, i+2$; $a = i+1$, $b = j+4$, and $c = i+3$ it follows that

$$V(1, i+1, i+2, j+4)_{i+1j+4} = \frac{L(1, i+2)\, V(1, i+1, i+2, i+3, j+4)_{i+1j+4}}{L(1, i+2, i+3)} - \frac{V(1, i+1, i+2, i+3)_{i+1i+3}\, V(1, i+2, i+3, j+4)_{i+3j+4}}{L(1, i+2, i+3)}. \tag{C6}$$

Substitution of (C4) into (C6) then gives

$$V(1, i+1, i+2, j+4)_{i+1j+4} = \frac{L(1, i+2)\, V(1, i+1, i+2, i+3, j+4)_{i+1j+4}}{L(1, i+2, i+3)} - \frac{\left(\begin{aligned} &V(1, i+2, i+3)_{i+2i+3}\, V(1, i+1, i+2, i+3)_{i+1i+3} \\ &\qquad \times V(1, i+2, i+3, j+4)_{i+2j+4}\end{aligned}\right)}{L(1, i+3)\, L(1, i+2, i+3)} - \frac{V(1, i+1, i+2, i+3)_{i+1i+3}\, V(1, i+3, j+4)_{i+3j+4}}{L(1, i+3)}. \tag{C7}$$

$V(1, i+2, j+4)_{i+2j+4}$

By (B9) with $\alpha = 1$; $a = i+2$, $b = j+4$, and $c = i+3$ it follows that

$$V(1, i+2, j+4)_{i+2j+4} = \frac{L(1)\, V(1, i+2, i+3, j+4)_{i+2j+4}}{L(1, i+3)} - \frac{V(1, i+2, i+3)_{i+2i+3}\, V(1, i+3, j+4)_{i+3j+4}}{L(1, i+3)}. \tag{C8}$$

The desired recurrence relations are expressed by (C5), (C7), and (C8). If the cofactors in these expressions are replaced by the *modified* cofactors of (19) and if use is made of (14), (15), and (18) these recurrence relations become those of (20a, b, c), respectively.

Appendix D

It is shown in this Appendix how (12), (13), and (17) can be reexpressed in terms of the *modified* cofactors of (19) and (20). Use is made of the identities of Appendix B.

$L(1, i, i+1, i+2) = 0$

By (B9) with $\alpha = 1$; $a = i$, $b = i+2$, and $c = i+1$

$$\begin{aligned} &V(1, i, i+1)_{ii+1}\, V(1, i+1, i+2)_{i+1i+2} \\ &\quad = L(1) V(1, i, i+1, i+2)_{ii+2} - L(1, i+1) V(1, i, i+2)_{ii+2} \\ &\quad = L(1) V(1, i, i+1, i+2)_{ii+2o} - L(1, i+1) V(1, i, i+2)_{ii+2o} \quad \text{by (B2)}, \end{aligned} \tag{D1}$$

$$\begin{aligned} \therefore\ &\frac{V(1, i, i+1, i+2)_{ii+2o}}{L(1, i+1)} \\ &\quad = \frac{V(1, i, i+2)_{ii+2o}}{L(1)} + \frac{V(1, i, i+1)_{ii+1}\, V(1, i+1, i+2)_{i+1i+2}}{L(1)\, L(1, i+1)} \\ &\quad = \frac{V(1, i, i+2)_{ii+2o}}{L(1)} - \frac{\bar{V}(1, i, i+2)_{ii+2}}{L(1)} \end{aligned} \tag{D2}$$

where the last step follows from (D1) and the definition of barred cofactors:

$$\bar{V}(1, i, i+2)_{ii+2} \equiv V(1, i, i+2)_{ii+2} \qquad \text{subject to } V(1, i, i+1, i+2)_{ii+2} = 0.$$

When the modified cofactors of (19) are introduced it is seen that

$$\begin{aligned} &\frac{V(1, i, i+1, i+2)_{ii+2o}}{L(1, i+1)} \\ &\quad = \frac{V(1, i, i+2)_{ii+2o}}{L(1)} - \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \bar{W}(1, i, i+2)_{ii+2}. \end{aligned} \tag{D3}$$

Then (22) results when this expression is substituted into (13).

$L(1, i, i+1, i+2, i+3) = 0$

By (B9) with $\alpha = 1, i+1$; $a = i$, $b = i+3$, and $c = i+2$

$$\begin{aligned} &V(1, i, i+1, i+2)_{ii+2}\, V(1, i+1, i+2, i+3)_{i+1i+3} \\ &\quad = L(1, i+1) V(1, i, i+1, i+2, i+3)_{ii+3} \\ &\qquad - L(1, i+1, i+2) V(1, i, i+1, i+3)_{ii+3} \\ &\quad = L(1, i+1) V(1, i, i+1, i+2, i+3)_{ii+3o} \\ &\qquad - L(1, i+1, i+2) V(1, i, i+1, i+3)_{ii+3o} \qquad \text{by (B2),} \end{aligned} \tag{D4}$$

$$\begin{aligned} &\therefore \frac{V(1, i, i+1, i+2, i+3)_{ii+3o}}{L(1, i+1, i+2)} \\ &= \frac{V(1, i, i+1, i+3)_{ii+3o}}{L(1, i+1)} + \frac{V(1, i, i+1, i+2)_{ii+2}\, V(1, i+1, i+2, i+3)_{i+1i+3}}{L(1, i+1)\, L(1, i+1, i+2)} \\ &= \frac{V(1, i, i+1, i+3)_{ii+3o}}{L(1, i+1)} - \frac{\bar{V}(1, i, i+1, i+3)_{ii+3}}{L(1, i+1)} \end{aligned} \tag{D5}$$

where the last step follows from (D4) and the definition of barred cofactors:

$$\begin{aligned} &\bar{V}(1, i, i+1, i+3)_{ii+3} \\ &\quad \equiv V(1, i, i+1, i+3)_{ii+3} \qquad \text{subject to } V(1, i, i+1, i+2, i+3)_{ii+3} = 0. \end{aligned}$$

By (B9) with $\alpha = 1$; $a = i$, $b = i+3$, $c = i+1$ it follows by analogy to (D3) that

$$\begin{aligned} &\frac{V(1, i, i+1, i+3)_{ii+3o}}{L(1, i+1)} \\ &\quad = \frac{V(1, i, i+3)_{ii+3o}}{L(1)} - \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \bar{W}(1, i, i+3)_{ii+3}. \end{aligned} \tag{D6}$$

Substituting (D6) into (D5) and introducing the modified cofactors of (19b) leads to

$$\frac{V(1, i, i+1, i+2, i+3)_{ii+3o}}{L(1, i+1, i+2)} = \frac{V(1, i, i+3)_{ii+3o}}{L(1)} - \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \overline{W}(1, i, i+3)_{ii+3} + \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \left[\frac{L(1)\, L(1, i, i+1)}{L(1, i)\, L(1, i+1)}\right]^{1/2} \overline{W}(1, i, i+1, i+3)_{ii+3}\,. \quad \text{(D7)}$$

Finally (23) results upon substituting this expression and (15) into (17).

$V(1, i, i+1, i+2, i+3, j+4)_{ij+4} = 0$

By (B9) with $\alpha = 1, i+1, i+2$; $a = i$, $b = j+4$, and $c = i+3$

$$\begin{aligned} &V(1, i, i+1, i+2, i+3)_{ii+3}\, V(1, i+1, i+2, i+3, j+4)_{i+1j+4} \\ &\quad = L(1, i+1, i+2) V(1, i, i+1, i+2, i+3, j+4)_{ij+4} \\ &\qquad - L(1, i+1, i+2, i+3) V(1, i, i+1, i+2, j+4)_{ij+4} \\ &\quad = L(1, i+1, i+2) V(1, i, i+1, i+2, i+3, j+4)_{ij+4o} \\ &\qquad - L(1, i+1, i+2, i+3) V(1, i, i+1, i+2, j+4)_{ij+4o} \qquad \text{by (B2)} \end{aligned} \quad \text{(D8)}$$

$$\begin{aligned} \therefore\ &\frac{V(1, i, i+1, i+2, i+3, j+4)_{ij+4o}}{L(1, i+1, i+2, i+3)} \\ &= \frac{V(1, i, i+1, i+2, j+4)_{ij+4o}}{L(1, i+1, i+2)} + \frac{V(1, i, i+1, i+2, i+3)_{ii+3}\, V(1, i+1, i+2, i+3, j+4)_{i+1j+4}}{L(1, i+1, i+2)\, L(1, i+1, i+2, i+3)} \\ &= \frac{V(1, i, i+1, i+2, j+4)_{ij+4o}}{L(1, i+1, i+2)} - \frac{V(1, i, i+1, i+2, j+4)_{ij+4}}{L(1, i+1, i+2)} \end{aligned} \quad \text{(D9)}$$

making use of (D8) and the physical requirement (6e) that $V(1, i, i+1, i+2, i+3, j+4)_{ij+4} = 0$.

By (B9) with $\alpha = 1, i+1$; $a = i$, $b = j+4$, $c = i+2$; it follows by analogy to (D4)–(D7) that

$$\frac{V(1, i, i+1, i+2, j+4)_{ij+4o}}{L(1, i+1, i+2)} = \frac{V(1, i, j+4)_{ij+4o}}{L(1)} - \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \overline{W}(1, i, j+4)_{ij+4} + \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \left[\frac{L(1)\, L(1, i, i+1)}{L(1, i)\, L(1, i+1)}\right]^{1/2} \overline{W}(1, i, i+1, j+4)_{ij+4}\,. \quad \text{(D10)}$$

Substituting (D10) into (D9) and introducing the modified cofactors of (19a) leads to

$$\frac{V(1, i, i+1, i+2, i+3, j+4)_{ij+4o}}{L(1, i+1, i+2, i+3)}$$
$$= \frac{V(1, i, j+4)_{ij+4o}}{L(1)} - \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \overline{W}(1, i, j+4)_{ij+4}$$
$$+ \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \left[\frac{L(1)\, L(1, i, i+1)}{L(1, i)\, L(1, i+1)}\right]^{1/2} \overline{W}(1, i, i+1, j+4)_{ij+4}$$
$$- \left[-\frac{L(1, i)}{L(1)}\right]^{1/2} \left[\frac{L(1)\, L(1, i, i+1)}{L(1, i)\, L(1, i+1)}\right]^{1/2}$$
$$\times \left[\frac{L(1, i+1)\, L(1, i, i+1, i+2)}{L(1, i, i+1)\, L(1, i+1, i+2)}\right]^{1/2} W(1, i, i+1, i+2, j+4)_{ij+4}\,.$$

When this is substituted into (12) and use is made of (15) the desired equation (24) is obtained.

Appendix E

Certain quantities of possible physical interest are expressed below in terms of invariant variables. This demonstration is made simply to indicate the usefulness of these variables. The quantities considered are (a) Treiman–Yang angles, (b) Toller angles, and (c) center of mass longitudinal and transverse momenta.

It will prove convenient in each of the following cases to make use of the identity [20].

$$e_{\mu\alpha\beta\gamma}(k_1)_\alpha(k_2)_\beta(k_3)_\gamma\, e_{\mu\nu\sigma\tau}(q_1)_\nu(q_2)_\sigma(q_3)_\tau = -\begin{vmatrix} k_1\cdot q_1 & k_1\cdot q_2 & k_1\cdot q_3 \\ k_2\cdot q_1 & k_2\cdot q_2 & k_2\cdot q_3 \\ k_3\cdot q_1 & k_3\cdot q_2 & k_3\cdot q_3 \end{vmatrix} \tag{E1}$$

where $e_{\mu\alpha\beta\gamma}$ is the totally antisymmetric symbol ($e_{0123} = +1$) and repeated 4-vector Greek indices are summed over using the Lorentz metric $g_{\mu\nu} = \mathrm{diag}\,[1, -1, -1, -1]$.

(a) *Treiman–Yang Angles*

The angle ϕ_{i+1} introduced in Eq. (13) of the text is actually a Treiman–Yang angle [21]. The physical significance of this angle is such that if the differential cross section in ϕ_{i+1} is flat then there is no dynamical correlation between particles in the set $\{3, 4,..., i+1\}$ and those in the set $\{i+2, i+3,..., n\}$. In the context of the multiperipheral model this means that no particle carrying spin is exchanged between particle $i+1$ and particle $i+2$.

From Eq. (14)

$$\cos\phi_{i+1} = -V(1, i, i+1, i+2)_{ii+2}/[L(1, i, i+1)L(1, i+1, i+2)]^{1/2} \tag{E2}$$

although this is never needed since values of ϕ_{i+1} are generated randomly during the course of the evaluation of the phase space integral. For illustrative purposes (E2) will be derived using the identity (E1).

Recall from the text that

$$\cos \phi_{i+1} = (\mathbf{p}_1 \times \mathbf{p}_{i+3n}) \cdot (\mathbf{p}_1 \times \mathbf{p}_{i+1}) / (|\mathbf{p}_1 \times \mathbf{p}_{i+3n}|\,|\mathbf{p}_1 \times \mathbf{p}_{i+1}|)$$

in the frame $\mathbf{p}_{i+2n} = 0$.

It is straightforward to show that this may be written in invariant form as

$$\cos \phi_{i+1} = -e_{\mu\alpha\beta\gamma}(p_1)_\alpha(p_{i+3n})_\beta(p_{i+2n})_\gamma\, e_{\mu\nu\sigma\tau}(p_1)_\nu(p_{i+1})_\sigma(p_{i+2n})_\tau \\ /\{|e_{\mu\alpha\beta\gamma}(p_1)_\alpha(p_{i+3n})_\beta(p_{i+2n})_\gamma|\,|e_{\mu\nu\sigma\tau}(p_1)_\nu(p_{i+1})_\sigma(p_{i+2n})_\tau|\}.$$

By using (E1) as well as the relations $p_{i+3n} = -p_{1i+2}$, $p_{i+2n} = -p_{1i+1}$, and $p_{i+1} = p_{1i+1} - p_{1i}$ it follows that

$$\cos \phi_{i+1} = -\{\text{cofactor of } p_{1i} \cdot p_{1i+2} \text{ in } \Delta(p_i, p_{1i}, p_{1i+1}, p_{1i+2})\} \\ /[\Delta(p_1, p_{1i}, p_{1i+1})\Delta(p_1, p_{1i+1}, p_{1i+2})]^{1/2}$$

in terms of Gram determinants. This is just (E2).

(b) *Toller Angles*

The Toller angle ω_i may be defined [10, 20] as the angle between the normals to the planes of $\mathbf{q}_1$, $\mathbf{p}_{i+1}$, and of $\mathbf{q}_2$, $\mathbf{p}_{i-1}$ in the frame $\mathbf{p}_i = 0$, where $\mathbf{q}_1 \equiv \bar{\mathbf{p}}_1 - \bar{\mathbf{p}}_{i+2n} = \mathbf{p}_{2i+1}$ and $\mathbf{q}_2 \equiv \bar{\mathbf{p}}_2 - \mathbf{p}_{3i-2} = -\mathbf{p}_{2i-2}$. Specifically $\cos \omega_i = -(\mathbf{p}_{2i+1} \times \mathbf{p}_{i+1}) \cdot (\mathbf{p}_{2i-2} \times \mathbf{p}_{i-1}) / (|\mathbf{p}_{2i+1} \times \mathbf{p}_{i+1}|\,|\mathbf{p}_{2i-2} \times \mathbf{p}_{i-1}|)$ in the frame $\mathbf{p}_i = 0$.
The dependence of a multiperipheral amplitude on ω_i signifies that particles with nonzero spin have been exchanged between particles $i - 1$ and i and/or between particles i and $i + 1$. That is, there is a correlation among the momenta of particles $i - 1$, i, and $i + 1$.

Putting the above expression in invariant form yields

$$\cos \omega_i = e_{\mu\alpha\beta\gamma}(p_{2i+1})_\alpha(p_{i+1})_\beta(p_i)_\gamma\, e_{\mu\nu\sigma\tau}(p_{2i-2})_\nu(p_{i-1})_\mu(p_i)_\gamma \\ /\{|e_{\mu\alpha\beta\gamma}(p_{2i+1})_\alpha(p_{i+1})_\beta(p_i)_\gamma|\,|e_{\mu\nu\sigma\tau}(p_{2i-2})_\nu(p_{i-1})_\beta(p_i)_\gamma|\}.$$

Using (E1) as well as the relations $p_{i+1} = p_{2i+1} - p_{2i}$, $p_i = p_{2i} - p_{2i-1}$, and $p_{i-1} = p_{2i-1} - p_{2i-2}$ it follows that

$$\cos \omega_i = -\{\text{cofactor of } p_{2i-2} \cdot p_{2i+1} \text{ in } \Delta(p_{2i-2}, p_{2i-1}, p_{2i}, p_{2i+1})\} \\ /[\Delta(p_{2i-2}, p_{2i-1}, p_{2i})\Delta(p_{2i-1}, p_{2i}, p_{2i+1})]^{1/2}$$

in terms of Gram determinants.

Notice that these Gram determinants are constructed from the vectors p_{2j}. They have no simple relationship to the Gram determinants used in the text which were formed from the vectors p_{1i}. Nevertheless the elements of the Gram determinants

(c) *Center of Mass Longitudinal and Transverse Momenta*

In the center of mass frame let θ_i be the angle between $\mathbf{p}_i$ and $\bar{\mathbf{p}}_1$ ($= -\bar{\mathbf{p}}_2 = -\mathbf{p}_1$) for $i = 3, 4, \ldots, n$. Then the longitudinal and transverse components of $\mathbf{p}_i$ are, respectively,

$$(\mathbf{p}_i)_\parallel \equiv |\,\mathbf{p}_i\,| \cos \theta_i \qquad \text{and} \qquad (\mathbf{p}_i)_\perp \equiv |\,\mathbf{p}_i\,| \sin \theta_i\,.$$

These may be put in terms of invariants by using the following relations

$$\begin{aligned} |\,\mathbf{p}_i\,|^2 &= (p_i)_0^2 - p_1 \cdot p_1 \\ &= [p_{12} \cdot p_i/(p_{12})_o]^2 - p_1 \cdot p_1 \\ &= [(p_{12} \cdot p_i)^2 - p_1{}^2 p_{12}^2]/p_{12}^2 \\ &= -\Delta(p_{12}\,, p_i)/\Delta(p_{12}) \end{aligned} \tag{E3}$$

where use has been made of the fact that $\mathbf{p}_{12} = 0$.

In a similar manner

$$\begin{aligned} \mathbf{p}_1 \cdot \mathbf{p}_i &= (p_1)_o\,(p_i)_o - p_1 \cdot p_i \\ &= [p_{12} \cdot p_1/(p_{12})_o][p_{12} \cdot p_i/(p_{12})_o] - p_1 \cdot p_i \\ &= [(p_{12} \cdot p_1)(p_{12} \cdot p_i) - p_{12}^2(p_1 \cdot p_i)]/p_{12}^2 \\ &= \{\text{cofactor of } p_1 \cdot p_i \text{ in } \Delta(p_1\,, p_{12}\,, p_i)\}/\Delta(p_{12}). \end{aligned} \tag{E4}$$

Thus, from (E3) and (E4)

$$\begin{aligned} (\mathbf{p}_i)_\parallel = -\mathbf{p}_1 \cdot \mathbf{p}_i/|\,\mathbf{p}_1\,| = &-\{\text{cofactor of } p_1 \cdot p_i \text{ in } \Delta(p_1\,, p_{12}\,, p_i)\} \\ &/[-\Delta(p_1\,, p_{12})\Delta(p_{12})]^{1/2}. \end{aligned}$$

Finally, by applying Eq. (A6) to $\Delta(p_1, p_{12}, p_i)$ and its cofactors to obtain $\sin \theta_i$ from $\cos \theta_i$,

$$\sin \theta_i = [\Delta(p_{12})\Delta(p_1\,, p_{12}\,, p_i)/\Delta(p_{12}\,, p_1)\Delta(p_{12}\,, p_i)]^{1/2}$$

it follows that

$$(\mathbf{p}_i)_\perp = [-\Delta(p_1\,, p_{12}\,, p_i)/\Delta(p_{12}\,, p_1)]^{1/2}.$$

Again, all elements of these Gram determinants may be put in terms of the Mandelstam invariants without difficulty. For example, $2p_{12} \cdot p_i =$

$$\begin{aligned} 2p_{12} \cdot (p_{3i} - p_{3i-i}) &= [(p_{12} + p_{3i})^2 - p_{12}^2 - p_{3i}^2] - [(p_{12} + p_{3i-1})^2 - p_{12}^2 - p_{3i-1}^2] \\ &= s_{1i} - s_{3i} - s_{1i-1} + s_{3i-1}\,. \end{aligned}$$

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